

THE GAMMA STEIN EQUATION AND NON-CENTRAL DE JONG THEOREMS

DEDICATED TO THE MEMORY OF CHARLES M. STEIN

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ABSTRACT. We study the Stein equation associated with the one-dimensional Gamma distribution, and provide novel bounds, allowing one to effectively deal with test functions supported by the whole real line. We apply our estimates to derive new quantitative results involving random variables that are non-linear functionals of random fields, namely: (i) a non-central quantitative de Jong theorem for sequences of degenerate U -statistics satisfying minimal uniform integrability conditions, significantly extending previous findings by de Jong (1990), Nourdin, Peccati and Reinert (2010) and Döbler and Peccati (2016), (ii) a new Gamma approximation bound on the Poisson space, refining previous estimates by Peccati and Thäle (2013), and (iii) new Gamma bounds on a Gaussian space, strengthening estimates by Nourdin and Peccati (2009). As a by-product of our analysis, we also deduce a new inequality for Gamma approximations *via* exchangeable pairs, that is of independent interest.

1. INTRODUCTION

1.1. Overview. The aim of this paper is to derive new explicit estimates for one-dimensional Gamma approximations, and then to apply our general findings to derive several non-central approximation results for sequences of random variables that have the form of non-linear functionals of a random measure. The random measures we are interested in are either the empirical measure associated with a sequence of independent random variables, or a Poisson or Gaussian measure. As discussed below, our applications significantly refine and generalise previous results about the Gamma approximation of degenerate and non necessarily symmetric U -statistics [DP16, dJ87, dJ89, dJ90, PT13], of smooth random variables on the Poisson space [PSTU10, PT13], and of smooth functionals of a Gaussian field [NPR10, NP09a, NP09b].

From now on, for fixed $r, \lambda \in (0, \infty)$, we will denote by $\Gamma(r, \lambda)$ the *Gamma distribution* with *shape parameter* r and *rate* λ which has probability density function (p.d.f.)

$$p_{r,\lambda}(x) = \begin{cases} \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{otherwise,} \end{cases}$$

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where

$$\Gamma(t) := \int_0^\infty x^{t-1} e^{-x} dx$$

denotes the *Euler Gamma function*. We denote the corresponding distribution function by $F_{r,\lambda}$. It is well-known that $X_{r,\lambda} \sim \Gamma(r, \lambda)$ has mean r/λ and variance r/λ^2 and that, if $Y = aX_{r,\lambda}$ for some $a > 0$, then Y has distribution $\Gamma(r, a^{-1}\lambda)$. For $\nu > 0$, we also denote by $\bar{\Gamma}(\nu)$ the so-called *centered Gamma distribution* with parameter ν which by definition is the distribution of

$$Z_\nu := 2X_{\nu/2,1} - \nu,$$

where, again, $X_{\nu/2,1}$ has distribution $\Gamma(\nu/2, 1)$. Notice that, if ν is an integer, then $\bar{\Gamma}(\nu)$ has a centered χ^2 distribution with ν degrees of freedom. According to the previous discussion, one has that

$$\mathbb{E}[Z_\nu] = 0 \quad \text{and} \quad \text{Var}(Z_\nu) = \mathbb{E}[Z_\nu^2] = 2\nu;$$

also, the following moment identity (already exploited in [NP09a]), will play an important role throughout the paper:

$$(1.1) \quad \mathbb{E}[Z_\nu^4] - 12\mathbb{E}[Z_\nu^3] - 12\nu^2 + 48\nu = 0.$$

One of our principal aims in the sections to follow is to obtain several explicit estimates on quantities of the type

$$d(W, X_{r,\lambda}) := \sup_{h \in \mathcal{H}} |\mathbb{E}[h(W)] - \mathbb{E}[h(X_{r,\lambda})]|,$$

where \mathcal{H} is a suitable class of test functions. The strategy we will adopt in order to do so, is to derive new estimates on the solutions of the *Gamma Stein equation*

$$(1.2) \quad xf'(x) + (r - \lambda x)f(x) = h(x) - \mathbb{E}[h(X_{r,\lambda})], \quad x \in \mathbb{R},$$

where h is an element of \mathcal{H} , and then to effectively use our bounds in the framework of *exchangeable pairs* (see [DP16, Ste86]). We will see that our results significantly extend the classical findings by [Luk94] and Pickett [Pic04], as well as the recent estimates from [GPR15]. In particular, one crucial feature of our approach is that we will be able to directly study the Stein equation (1.2) on the whole real line, although the target distribution $\Gamma(r, \lambda)$ is supported on the positive real axis. As discussed in Section (1.4), in the specific case of Gamma approximations on a Gaussian space, our results remarkably allow one to obtain quantitative limit theorems in the 1-Wasserstein distance (see below for definitions).

As anticipated, our main motivation comes from the study of the non-central fluctuations of random objects which can be expressed in terms of iterated stochastic integrals with respect to a given random measure. The next three subsections contain a detailed discussion of our main applications to degenerate U -statistics and multiple integrals on the Poisson and Gaussian spaces.

1.2. A non-central de Jong theorem. Let X_1, \dots, X_n be independent random variables on some generic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with values in arbitrary measurable spaces $(E_1, \mathcal{E}_1), \dots, (E_n, \mathcal{E}_n)$. In the recent paper [DP16] we were able to prove error bounds for the uni- and multivariate normal approximation of (vectors of) degenerate, non-symmetric U -statistics of the data vector $X = (X_1, \dots, X_n)$. In particular, we were able to provide a complete quantitative extension of a CLT by de Jong [dJ90] which roughly states that a normalized sequence W_n , $n \in \mathbb{N}$, of such U -statistics converges weakly to the standard normal distribution if the sequence of fourth moments converges to 3 and some asymptotic Lindberg-type condition is satisfied — see formula (1.7) below.

The main abstract results of the present paper are used to continue such a line of research by dealing with the approximation of such a degenerate, non-symmetric U -statistic by a centered Gamma distribution. More precisely, assume that

$$\psi : \prod_{j=1}^n E_j \rightarrow \mathbb{R} \quad \text{is} \quad \bigotimes_{j=1}^n \mathcal{E}_j - \mathcal{B}(\mathbb{R}) \text{ - measurable}$$

and that

$$W := \psi(X_1, \dots, X_n) \in L^4(\mathbb{P})$$

satisfies

$$(1.3) \quad \mathbb{E}[W] = 0 \quad \text{and} \quad \mathbb{E}[W^2] = 2\nu$$

for some $\nu > 0$. We write

$$[n] := \{1, \dots, n\}$$

and for $J \subseteq [n]$ we define

$$\mathcal{F}_J := \sigma(X_j, j \in J).$$

We denote by

$$(1.4) \quad W = \sum_{J \subseteq [n]} W_J$$

the *Hoeffding decomposition* of W (see e.g. [DP16, Hoe48, KR82, KB94, Ser80, Vit92]). Note that this means that, for each $J \subseteq [n]$, W_J is \mathcal{F}_J -measurable and that

$$\mathbb{E}[W_J | \mathcal{F}_K] = 0,$$

whenever $J \not\subseteq K$. It is well-known that W admits a Hoeffding decomposition of the type (1.4), as long as $W \in L^1(\mathbb{P})$ and that it is almost surely unique and given by

$$(1.5) \quad W_J = \sum_{L \subseteq J} (-1)^{|J|-|L|} \mathbb{E}[W | \mathcal{F}_L], \quad J \subseteq [n].$$

We can thus write

$$W_J = \psi_J(X_j, j \in J)$$

for some measurable function

$$\psi_J : \prod_{j \in J} E_j \rightarrow \mathbb{R}, \quad J \subseteq [n].$$

Let us also define

$$\sigma_J^2 := \text{Var}(W_J), \quad J \subseteq [n].$$

One major assumption in what follows will be that, for some fixed integer $d \in [n]$, W is a *degenerate U -statistic of order d* (or d -degenerate U -statistic), i.e. that the Hoeffding decomposition (1.4) has the form

$$(1.6) \quad W = \sum_{J \in \mathcal{D}_d} W_J,$$

where

$$\mathcal{D}_d := \{J \subseteq [n] : |J| = d\}$$

denotes the collection of all $\binom{n}{d}$ d -subsets of $[n]$, i.e., we assume that $W_K = 0$ \mathbb{P} -a.s. whenever $|K| \neq d$. Hence, we have

$$W = \psi(X_1, \dots, X_n) = \sum_{J \in \mathcal{D}_d} \psi_J(X_j, j \in J).$$

Furthermore, we define the quantities

$$\varrho^2 := \varrho_n^2 := \max_{1 \leq i \leq n} \sum_{\substack{K \in \mathcal{D}_d: \\ i \in K}} \sigma_K^2 \quad \text{and} \quad D := D_n := \max_{J \in \mathcal{D}_d} \frac{\mathbb{E}[W_J^4]}{\sigma_J^4}.$$

One of the main results of the present paper is an explicit upper bound on a certain probability distance between the law of W and $\bar{\Gamma}(\nu)$. For $k \in \mathbb{N}$, denote by \mathcal{H}_k the class of those $(k-1)$ -times differentiable test functions h on \mathbb{R} such that $h^{(k-1)}$ is Lipschitz-continuous and we have

$$\|h^{(l)}\|_\infty \leq 1 \quad \text{for } l = 1, \dots, k.$$

For real random variavbles X and Y such that $\mathbb{E}|X|, \mathbb{E}|Y| < \infty$ we denote by

$$d_k(X, Y) := d_k(\mathcal{L}(X), \mathcal{L}(Y)) := \sup_{h \in \mathcal{H}_k} |\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]|$$

the distance between the distributions of X and Y induced by the class \mathcal{H}_k ; observe that d_1 coincides with the classical 1-*Wasserstein distance*, see e.g. [NP12, Appendix C] and the references therein. The next theorem estimates the d_2 -distance between the law of W and $\bar{\Gamma}(\nu)$ in terms of the analogous linear combination of the moments of W as well as in terms of the quantities ϱ_n^2 and D_n .

THEOREM 1.1. *Under the above assumptions we have the bound*

$$\begin{aligned} d_2(W, Z_\nu) &\leq \frac{\max(1, \frac{2}{\nu})}{\sqrt{3}} \sqrt{|\mathbb{E}[W^4] - 12\mathbb{E}[W^3] - 12\nu^2 + 48\nu|} \\ &\quad + \frac{(2\sqrt{3} + 4\sqrt{\nu}) \max(1, \frac{2}{\nu}) + 4\sqrt{\nu}}{3\sqrt{d}} \sqrt{C_d D_n \varrho_n^2}, \end{aligned}$$

where C_d is a finite constant which only depends on d .

One should immediately notice that the factors

$$\frac{\max(1, \frac{2}{\nu})}{\sqrt{3}}, \quad \text{and} \quad \frac{(2\sqrt{3} + 4\sqrt{\nu}) \max(1, \frac{2}{\nu}) + 4\sqrt{\nu}}{3\sqrt{d}},$$

both diverge to infinity as $\nu \rightarrow 0$. As formally discussed in Remark 2.8, this somewhat undesirable feature seems to be unavoidable: in particular, such a phenomenon is

related to the fact that, for our applications, we need to be able to deal with random variables whose distribution is possibly supported by the whole real line.

The estimate in Theorem 1.1 immediately yields the following limit result.

COROLLARY 1.2. *Fix $\nu > 0$ and an integer $d \geq 1$ and let $\{n_m : m \geq 1\}$ be a sequence of integers diverging to infinity. Let $\{W_m : m \geq 1\}$ be a sequence of centered, degenerate U -statistics of order d with $\mathbb{E}[W_m^2] = 2\nu$, such that each W_m is a function of the vector of independent variables $(X_1^{(m)}, \dots, X_{n_m}^{(m)})$. Then, if*

$$\lim_{m \rightarrow \infty} (\mathbb{E}[W_m^4] - 12\mathbb{E}[W_m^3] - 12\nu^2 + 48\nu) = 0 = \lim_{m \rightarrow \infty} D_{n_m} \varrho_{n_m}^2,$$

the sequence $\{W_m : m \geq 1\}$ converges in distribution to Z_ν .

Plainly, the asymptotic relation $\lim_{m \rightarrow \infty} D_{n_m} \varrho_{n_m}^2 = 0$ is verified whenever the sequence $\{D_{n_m}\}$ is bounded, and $\varrho_{n_m}^2 \rightarrow 0$; see the discussion below.

It is also instructive to compare Theorem 1.1 and Corollary 1.2 with the main findings of [DP16], applying to the case where the assumption $\mathbb{E}[W^2] = 2\nu$ in (1.3) is replaced by $\mathbb{E}[W^2] = 1$. In this framework, letting Z be a standard normal random variable, one deduces from [DP16, Theorem 1.3] that

$$(1.7) \quad d_1(W, Z) \leq \left(\sqrt{\frac{2}{\pi}} + \frac{4}{3} \right) \sqrt{|\mathbb{E}[W^4] - 3|} + \sqrt{\kappa_d} \left(\sqrt{\frac{2}{\pi}} + \frac{2\sqrt{2}}{\sqrt{3}} \right) \varrho_n,$$

where κ_d is a finite constant which only depends on d . As demonstrated in [DP16], from (1.7) one can immediately deduce de Jong's theorem [dJ90]: *Fix $d \geq 1$, and let $\{n_m : m \geq 1\}$ be a sequence of integers diverging to infinity. Let $\{W_m : m \geq 1\}$ be a sequence of unit variance degenerate U -statistics of order d , such that each W_m is a function of the vector of independent variables $(X_1^{(m)}, \dots, X_{n_m}^{(m)})$. Then, as $m \rightarrow \infty$, if $\mathbb{E}[W_m^4] \rightarrow 3$ and $\varrho_{n_m}^2 \rightarrow 0$, one has that W_m converges in distribution towards a standard Gaussian random variable.*

REMARK 1.3. (a) In view of relation (1.1), Corollary 1.2 is an analog of de Jong's theorem [dJ90] in the context of a Gamma limit.

(b) We conjecture that, analogously to the bounds on normal approximations derived in [DP16], the quantity D_n could be removed from the bound in Theorem 1.1 and, hence, also from the limit theorem stated in Corollary 1.2.

(c) In [NPR10] the authors prove an error bound on the centered Gamma approximation (for integer ν) of homogeneous multilinear forms in independent and normalized real-valued random variables $(X_i)_{i \in \mathbb{N}}$. These form a particularly important example class of degenerate, non-symmetric U -statistics. Their bound also involves the quantities $|\mathbb{E}[W^4] - 12\mathbb{E}[W^3] - 12\nu^2 + 48\nu|$, ϱ_n^2 and $\beta := \sup_{i \in \mathbb{N}} \mathbb{E}[X_i^4]$ and it is easy to see that the condition $\beta < \infty$ is in fact equivalent to the condition $\sup_{n \in \mathbb{N}} D_n < \infty$ in this special situation. Thus, Theorem 1.1 and Corollary 1.2 can be seen as an extension and improvement of the bounds and limit theorems from [NPR10] to a wider class of statistics.

1.3. Gamma limits on the Poisson space. In this subsection, we describe how our new bounds on the solution to the Gamma Stein equation (1.2), yield new analytic estimates for the Gamma approximation of functionals of a Poisson random measure. We will first briefly introduce the setup and some necessary notation. Further technical details are provided in Section 4. For any unexplained notions

we refer to the recent book [PR16], as well as to the existing related literature, e.g. [PSTU10, LRP13a, LRP13b, PT13]. We stress that limit theorems and probabilistic approximations involving non-linear functionals of a Poisson measure have gained enormous momentum in recent years, specially in connections with the large scale analysis of random geometric structures – see again [PR16], and the references therein.

We now fix a Polish space \mathcal{Z} as well as a σ -finite, non-atomic measure μ on the Borel- σ -field \mathcal{Z} on \mathcal{Z} . Furthermore, we let

$$\mathcal{Z}_\mu := \{B \in \mathcal{Z} : \mu(B) < \infty\}$$

and denote by

$$\eta = \{\eta(B) : B \in \mathcal{Z}_\mu\}$$

a *Poisson measure* on $(\mathcal{Z}, \mathcal{Z})$ with *control* μ , defined on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We recall that the distribution of η is completely determined by the following two facts: (i) for each pair of disjoint sets $B, C \in \mathcal{Z}_\mu$, the random variables $\eta(B)$ and $\eta(C)$ are independent, and (ii) that for every $B \in \mathcal{Z}_\mu$, the random variable $\eta(B)$ has the Poisson distribution with mean $\mu(B)$. For $B \in \mathcal{Z}_\mu$, we also write $\hat{\eta}(B) := \eta(B) - \mu(B)$ and denote by

$$\hat{\eta} = \{\hat{\eta}(B) : B \in \mathcal{Z}_\mu\}$$

the *compensated Poisson measure* associated with η . Without loss of generality, we may and will assume that $\mathcal{F} = \sigma(\eta)$.

Our main result in this section involves the following Malliavin operators: (i) the *Malliavin derivative* D , (ii) the *generator of the Ornstein-Uhlenbeck semigroup* L , and (iii) the *pseudo-inverse* of L , written L^{-1} . Formal definitions and details are provided in Section 4. Here, we only recall that the spectrum of L is given by the negative integers $\{-p : p = 0, 1, 2, \dots\}$ and that $F \in \text{Ker}(L + pI)$ (that is, F is an eigenfunction of L , with eigenvalue $-p$) if and only if $F = I_p(f)$, where I_p indicates a multiple Wiener-Itô integral of order p with respect to $\hat{\eta}$, and f is a suitable square-integrable kernel. The eigenspace $\text{Ker}(L + pI)$ is customarily called the p th *Wiener chaos* associated with η .

The next statement – whose proof exploits our new results on the solution to the Stein equation (1.2) – is our main estimate on the Poisson space: in particular, its second part contains the announced result for multiple Wiener-Itô integrals. Proofs are deferred to Section 4.

THEOREM 1.4. *Let $F \in L^2(\mathbb{P})$ be centered, and assume that F belongs to the domain of the Malliavin derivative operator D . Then, we have the bounds*

$$\begin{aligned} d_2(F, Z_\nu) &\leq \max\left(1, \frac{2}{\nu}\right) \mathbb{E} \left| 2(F + \nu) - \langle DF, -DL^{-1}F \rangle_{L^2(\mu)} \right| \\ &\quad + \max\left(1, \frac{1}{\nu} + \frac{1}{2}\right) \int_{\mathcal{Z}} \mathbb{E}[|D_z F|^2 | D_z L^{-1}F|] \mu(dz) \end{aligned} \tag{1.8}$$

$$\begin{aligned} &\leq \max\left(1, \frac{2}{\nu}\right) \sqrt{\mathbb{E} \left[\left(2(F + \nu) - \langle DF, -DL^{-1}F \rangle_{L^2(\mu)} \right)^2 \right]} \\ &\quad + \max\left(1, \frac{1}{\nu} + \frac{1}{2}\right) \int_{\mathcal{Z}} \mathbb{E}[|D_z F|^2 | D_z L^{-1}F|] \mu(dz). \end{aligned} \tag{1.9}$$

Here, we have used the standard notation

$$\langle DF, -DL^{-1}F \rangle_{L^2(\mu)} = - \int_{\mathcal{Z}} (D_z F) (D_z L^{-1} F) \mu(dz).$$

If, furthermore, $F = I_p(f)$ for some $p \geq 1$ and some square-integrable kernel f , then

$$\langle DF, -DL^{-1}F \rangle_{L^2(\mu)} = p^{-1} \|DF\|_{L^2(\mu)}^2 \quad \text{and}$$

$$\int_{\mathcal{Z}} \mathbb{E}[|D_z F|^2 |D_z L^{-1} F|] \mu(dz) = p^{-1} \int_{\mathcal{Z}} \mathbb{E}[|D_z F|^3] \mu(dz)$$

so that the previous estimates becomes

$$(1.10) \quad \begin{aligned} d_2(F, Z_\nu) &\leq \max\left(1, \frac{2}{\nu}\right) \mathbb{E} \left| 2(F + \nu) - p^{-1} \|DF\|_{L^2(\mu)}^2 \right| \\ &\quad + p^{-1} \max\left(1, \frac{1}{\nu} + \frac{1}{2}\right) \int_{\mathcal{Z}} \mathbb{E}[|D_z F|^3] \mu(dz) \end{aligned}$$

$$(1.11) \quad \begin{aligned} &\leq \max\left(1, \frac{2}{\nu}\right) \sqrt{\mathbb{E} \left[\left(2(F + \nu) - p^{-1} \|DF\|_{L^2(\mu)}^2 \right)^2 \right]} \\ &\quad + p^{-1} \max\left(1, \frac{1}{\nu} + \frac{1}{2}\right) \int_{\mathcal{Z}} \mathbb{E}[|D_z F|^3] \mu(dz). \end{aligned}$$

REMARK 1.5. The content of Theorem 1.4 should be directly compared with [PT13, Theorem 2.1], according to which

$$\begin{aligned} d_3(F, Z_\nu) &\leq c_1 \mathbb{E} \left| 2(F + \nu)_+ - \langle DF, -DL^{-1}F \rangle_{L^2(\mu)} \right| \\ &\quad + c_2 \int_{\mathcal{Z}} \mathbb{E}[|D_z F|^2 |D_z L^{-1} F|] \mu(dz) \\ &\quad + 2c_1 \int_{\mathcal{Z}} \mathbb{E}[(D_z F \mathbf{1}_{(F > -\nu)})(D_z F) |D_z L^{-1} F|] \mu(dz), \end{aligned}$$

where c_1, c_2 are explicit constants uniquely depending on ν . Note that our estimate (1.8) improves on such an estimate in three ways: (i) the distance d_3 is replaced by the less smooth distance d_2 , (ii) the first expectation on the right-hand side does not involve the positive part of $F + \nu$, and (iii) the third term in the bound has been completely removed. As will become evident in the proof, Points (i) and (iii) are a direct consequence of the fact that our approach allows us to solve and control equation (1.2) on the whole real line, thus obtaining more tractable solutions than those used in [PT13]. Note that our bound can be directly used to deduce simplified proofs of the other estimates proved in [PT13], like e.g. [PT13, Theorem 2.6 and Proposition 2.9]. Details are left to the reader.

1.4. Gamma limits on a Gaussian space. We conclude this section by showing how the results of the present paper can also be used to give better estimates on the Gamma approximation of non-linear functionals of Gaussian fields, thus improving results from [NP09b, NP13]. For the sake of conciseness, in this section we will keep explicit definitions to a minimum, and refer the reader to the monograph [NP12] for any unexplained notion or detail.

Now let \mathcal{H} be a real separable Hilbert space, and let $X = \{X(h) : h \in \mathcal{H}\}$ be an *isonormal Gaussian process* over \mathcal{H} . We assume that X is defined on a suitable

probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and that $\mathcal{F} = \sigma(X)$. Similarly to the previous section, we associate to X the following canonical Malliavin operators: (i) the *Malliavin derivative* D (whose domain is indicated by $\mathbb{D}^{1,2}$), (ii) the *generator of the Ornstein-Uhlenbeck semigroup* L , and (iii) the *pseudo-inverse* of L , written again L^{-1} . As on the Poisson space, the spectrum of L is given by the negative integers $\{-p : p = 0, 1, 2, \dots\}$ and one has that $F \in \text{Ker}(L + pI)$ (that is, the p th *Wiener chaos* of X) if and only if $F = I_p(f)$, where I_p indicates a multiple Wiener-Itô integral of order p , and f is an element of the symmetric tensor product $\mathcal{H}^{\odot p}$.

One has the following estimate (recall that d_1 corresponds to the 1-Wasserstein distance).

THEOREM 1.6. *Let F be centered element of $\mathbb{D}^{1,2}$ and fix $\nu > 0$. Then,*

$$(1.12) \quad d_1(F, Z_\nu) \leq \max\left(1, \frac{2}{\nu}\right) \mathbb{E} \left| \mathbb{E} \{ 2(F + \nu) - \langle DF, -DL^{-1}F \rangle_{\mathcal{H}} \mid F \} \right| \\ \leq \max\left(1, \frac{2}{\nu}\right) \mathbb{E} \left[\left(2(F + \nu) - \langle DF, -DL^{-1}F \rangle_{\mathcal{H}} \right)^2 \right]^{1/2}.$$

If $F \in \text{Ker}(L + pI)$ for some integer $p \geq 2$, then the previous estimate becomes

$$(1.13) \quad d_1(F, Z_\nu) \leq \max\left(1, \frac{2}{\nu}\right) \mathbb{E} \left| \mathbb{E} \{ 2(F + \nu) - p^{-1} \|DF\|_{\mathcal{H}}^2 \mid F \} \right|.$$

Inequality (1.12) improves [NP09b, Theorem 3.11], where a similar upper bound is proved for a smoother distance (written $d_{\mathcal{H}_2}$ therein) involving test functions of class C^2 with bounded derivatives. By inspection of the proofs contained in [NP09b], one sees that such a smoothness requirement on test functions is indeed an artefact of the bounds contained in [Luk94]. By combining Theorem 1.6 with the main findings from [NP13] and with some computations from [APP15], one also obtains the following non-trivial quantitative characterisation of Gamma convergence in total variation inside a fixed sum of Wiener chaoses. We recall that, given two real-valued random variables X, Y , the *total variation distance* between the distributions of X and Y is given by

$$d_{TV}(X, Y) = \sup_{A \in \mathcal{B}(\mathbb{R})} |\mathbb{P}[X \in A] - \mathbb{P}[Y \in A]|,$$

where $\mathcal{B}(\mathbb{R})$ stands for the class of all Borel subsets of \mathbb{R} .

PROPOSITION 1.7. *Fix $\nu > 0$, as well as an integer $m \geq 2$, and let $\{F_n : n \geq 1\} \subseteq \bigoplus_{p=1}^m \text{Ker}(L + pI)$ be such that $\mathbb{E}[F_n^2] \rightarrow 2\nu$. Then, F_n converges in distribution to Z_ν if and only if*

$$(1.14) \quad \mathbb{E} \left| \mathbb{E} \{ 2(F_n + \nu) - \langle DF_n, -DL^{-1}F_n \rangle_{\mathcal{H}} \mid F_n \} \right| \rightarrow 0, \quad n \rightarrow \infty,$$

and there exists a finite constant $c > 0$ (not depending on n) such that

$$(1.15) \quad d_{TV}(F_n, Z_\nu) \leq c \left(\mathbb{E} \left| \mathbb{E} \{ 2(F_n + \nu) - \langle DF_n, -DL^{-1}F_n \rangle_{\mathcal{H}} \mid F_n \} \right| \right)^{1/2m+1}.$$

One has also to observe that, according to [NP09a], if the sequence $\{F_n\}$ in Proposition 1.7 is such that $\{F_n\} \subseteq \text{Ker}(L + mI)$ and (1.14) is verified, then necessarily m is an even integer. See also [AS15, KTar] for some related limit theorems. The proofs of Theorem 1.6 and Proposition 1.7 are given in Section 5.

2. STEIN'S METHOD AND EXCHANGEABLE PAIRS FOR GAMMA APPROXIMATIONS

2.1. Main estimates for Gamma approximations. *Stein's method* is a popular technique for estimating the distance between the distribution of some given random variable W and a usually better understood target distribution. It was first developed by Stein [Ste72] for the standard normal distribution and has by now been extended to many other univariate distributions, like the Poisson (see e.g. [Che75], [AGG89] or [BHJ92]), the Exponential (see e.g. [CFR11], [PR11] and [FR13]), the Beta ([GR13] and [Döb15]), the Gamma ([Luk94], [Pic04], [Gau13] and [GPR15]) and the Variance-Gamma (see [Gau14]) distributions.

Stein's method for the Gamma distribution was first considered by Luk [Luk94]. There it was found that a real random variable X has the $\Gamma(r, \lambda)$ distribution if and only if

$$\mathbb{E}[Xf'(X)] = -\mathbb{E}[(r - \lambda X)f(X)]$$

holds for a sufficiently rich class of functions f . Following Stein's seminal idea this led him to the Gamma Stein equation (1.2), which, given the test function h on \mathbb{R} with $\mathbb{E}|h(X_{r,\lambda})| < \infty$, is to be solved for f . Usually, this equation is only considered and solved on the support $[0, \infty)$ of $\Gamma(r, \lambda)$ but for our purposes we will need a solution f_h to (1.2) which is defined on the whole real line. Here, by a solution of (1.2) we mean a function f on \mathbb{R} which is locally absolutely continuous and which satisfies (1.2) at those points at which it is in fact differentiable. Given such a function, contrary to the usual convention, we define f' at the non-zero points of non-differentiability of f by (1.2). If f is not differentiable at 0, then, for definiteness, we let $f'(0) := 0$. For a test function h as above, a solution f_h to (1.2) and a given real-valued random variable W we thus obtain

$$(2.1) \quad \left| \mathbb{E}[h(W)] - \mathbb{E}[h(X_{r,\lambda})] \right| = \left| \mathbb{E}[Wf'_h(W) + (r - \lambda W)f_h(W)] \right|,$$

whenever the right hand side is well-defined. As it turns out, the right hand side of (2.1) may often be efficiently bounded by means of some additional tool exploiting the structure of the random quantity W . This might be a similar characterization for the law of W , an integration by parts formula on the space where W is defined, or a suitable coupling construction.

In any case, in order to bound the right hand side of (2.1) it is crucial to have smoothness bounds on the solution f_h of (1.2) in terms of the test function h . One of the theoretical contributions of this paper is to provide a new set of such bounds which are valid for the solution f_h on the whole real line, not just on $[0, \infty)$. This is essential for our purposes, as the random variables W we consider in our applications do not necessarily have range included in the positive axis. Another consequence of our new bounds is an improvement of Theorem 2.1 from [PT13] and its consequences which deals with the Gamma approximation of functionals of a Poisson random measure.

To deal with our main application in this paper, we develop the technique of *exchangeable pairs* in the context of Gamma approximation. This coupling construction lies at the heart of Stein's method and was first considered for normal approximation in Stein's celebrated monograph [Ste86]. In the recent paper [DP16] the authors

applied it to the uni- and multivariate approximation of (vectors of) degenerate U -statistics. In particular, we were able to derive a complete quantitative extension of a famous CLT by de Jong [dJ90].

In what follows, for a function f on \mathbb{R} , we denote by

$$\|f'\|_\infty := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \in [0, \infty) \cup \{+\infty\}$$

its minimum Lipschitz constant. This notation does not cause any confusion as it coincides with the supremum norm of the derivative of f whenever f is differentiable. Similarly, if f is n -times differentiable for some $n \geq 1$, we denote by $\|f^{(n+1)}\|_\infty$ the minimum Lipschitz constant of $f^{(n)}$. We can now state our new smoothness estimates for the solution f_h of (1.2) on \mathbb{R} . We defer the proof of the next theorem to the end of this section.

THEOREM 2.1. (a) *Let h be Lipschitz-continuous on \mathbb{R} . Then, there exists a Lipschitz-continuous solution f_h of (1.2) on \mathbb{R} which satisfies the bounds*

$$(2.2) \quad \|f_h\|_\infty \leq \lambda^{-1} \|h'\|_\infty \quad \text{and} \quad \|f'_h\|_\infty \leq 2 \max\left(1, \frac{1}{r}\right) \|h'\|_\infty.$$

(b) *Suppose that h is continuously differentiable on \mathbb{R} and that both h and h' are Lipschitz-continuous. Then, the solution f_h of (1.2) from (a) is continuously differentiable and its derivative f'_h is Lipschitz-continuous with minimum Lipschitz constant*

$$(2.3) \quad \|f''_h\|_\infty \leq 4\lambda \max\left(1, \frac{1}{r}\right) \|h'\|_\infty + 2\|h''\|_\infty.$$

REMARK 2.2. (a) By inspection of the proof of Theorem 2.1, one sees that the following refinement of (2.2) holds: writing f_h^+ and f_h^- for the restriction of f_h to \mathbb{R}_+ and \mathbb{R}_- , respectively, one has that

$$(2.4) \quad \|(f_h^+)' \|_\infty \leq 2\|h'\|_\infty, \quad \text{and} \quad \|(f_h^-)' \|_\infty \leq \frac{2}{r} \|h'\|_\infty$$

We will see in Remark 2.8 that, in principle, the quantity $2/r$ in the previous estimate *cannot* be replaced by a factor that is uniformly bounded in r .

(b) Using the iterative technique for bounding higher derivatives of solutions to Stein equations from [Döb15] which is further detailed in the recent paper [DGV15], from the bound given in Theorem 2.1 (b), we can easily derive the bound

$$(2.5) \quad \begin{aligned} \|f_h^{(k)}\|_\infty &\leq 2^k \lambda^{k-1} (k-1)! \max\left(1, \frac{1}{r}\right) \|h'\|_\infty \\ &\quad + \sum_{j=0}^{k-2} 2^{j+1} \lambda^j \frac{(k-1)!}{(k-j-1)!} \|h^{(k-j)}\|_\infty, \end{aligned}$$

valid for each $k \geq 1$ and each $(k-1)$ -times differentiable test function h , whose first $k-1$ derivatives are bounded and whose $(k-1)$ st derivative is Lipschitz-continuous.

(c) In the recent paper [GPR15], the authors proved that for each $k \geq 1$ and each test function h from some specific sub-class $\mathcal{C}_{\lambda,k}$ of all $(k-1)$ -times differentiable

functions such that $h^{(k-1)}$ is still absolutely continuous, the following bound holds:

$$(2.6) \quad \sup_{x>0} |f_h^{(k)}(x)| \leq \frac{2}{r+k} \left(3 \sup_{x>0} |h^{(k)}(x)| + 2\lambda \sup_{x>0} |h^{(k-1)}(x)| \right)$$

Note that, as opposed to the bounds from Theorem 2.1 or the bound (2.5), this bound converges to 0 whenever the shape parameter r of the Gamma distribution goes to ∞ , which can be beneficial for certain applications as demonstrated in [GPR15]. However, the bounds given in the present paper are valid on the whole real line and are thus applicable to a broader class of applications. We conjecture that there do exist positive, finite constants $C_{r,\lambda}^{(1)}$ and $C_{r,\lambda}^{(2)}$ with

$$\|f_h'\|_\infty \leq C_{r,\lambda}^{(1)} \|h'\|_\infty \quad \text{and} \quad \|f_h''\|_\infty \leq C_{r,\lambda}^{(2)} (\|h'\|_\infty + \|h''\|_\infty)$$

such that $\lim_{r \rightarrow \infty} C_{r,\lambda}^{(1)} = \lim_{r \rightarrow \infty} C_{r,\lambda}^{(2)} = 0$. These may be derived by a more careful investigation of the solutions f_h on the support interval $[0, \infty)$. On the other hand, as already mentioned (see again Remark 2.8), the property $\lim_{r \downarrow 0} C_{r,\lambda}^{(1)} = \infty$ is inevitable, as opposed to the bounds (2.6) for the solutions on $(0, \infty)$.

2.2. Targeting the centered Gamma distribution. Next, we transfer the bounds found in Theorem 2.1 to the centered Gamma distribution $\bar{\Gamma}(\nu)$ of Z_ν and state an off-the-shelf result, which bounds the distance between the distribution of a given random variable W and $\bar{\Gamma}(\nu)$ in terms of an exchangeable pair. To the best of our knowledge this approach has not been considered in the context of Gamma approximation so far. The Stein equation for $\bar{\Gamma}(\nu)$ we use is given by

$$(2.7) \quad 2(x + \nu)f'(x) - xf(x) = h(x) - \mathbb{E}[h(Z_\nu)],$$

where h is Borel-measurable on \mathbb{R} with $\mathbb{E}|h(Z_\nu)| < \infty$.

THEOREM 2.3. (a) *Let h be Lipschitz-continuous on \mathbb{R} . Then, there exists a Lipschitz-continuous solution f_h of (2.7) on \mathbb{R} which satisfies the bounds*

$$\|f_h\|_\infty \leq \|h'\|_\infty \quad \text{and} \quad \|f_h'\|_\infty \leq \max\left(1, \frac{2}{\nu}\right) \|h'\|_\infty.$$

(b) *Suppose that h is continuously differentiable on \mathbb{R} such that both h and h' are Lipschitz-continuous. Then, there is a continuously differentiable solution f_h of (2.7) on \mathbb{R} whose derivative f_h' is Lipschitz-continuous with minimum Lipschitz constant*

$$\|f_h''\|_\infty \leq \max\left(1, \frac{2}{\nu}\right) \|h'\|_\infty + \|h''\|_\infty.$$

REMARK 2.4. (a) Plainly, refined bounds analogous to (2.4) can be obtained for the function f_h appearing in the statement of Theorem 2.3.

(b) In [PT13] the slightly different Stein equation

$$(2.8) \quad 2(x + \nu)_+ f'(x) - xf(x) = h(x) - \mathbb{E}[h(Z_\nu)]$$

with $g_+ := \max(g, 0)$ was used. It turns out that the solution f_h of (2.7) from Theorem 2.3 has better smoothness properties at the singularity point $x = -\nu$ of the Stein equation than the solution of (2.8) considered in [PT13]. This makes it possible for us to improve the bounds on Gamma approximation on the Poisson space provided there. Furthermore, for the application to U -statistics in the

present paper, it is essential to have a linear coefficient function for f' in the Stein equation. This will become clear from the proof of Theorem 1.1 in Section 3.

Proof of Theorem 2.3. Given h with $\mathbb{E}|h(Z_\nu)| < \infty$ we define h_1 by $h_1(x) := h(2x - \nu)$. It is easy to see that if g_h is the solution to (1.2) with h replaced by h_1 from Theorem 2.1, then the function f_h with

$$f_h(x) := \frac{1}{2}g_h\left(\frac{x + \nu}{2}\right)$$

solves (2.7). Furthermore, from Theorem 2.1 we obtain the bounds

$$\begin{aligned} \|f_h\|_\infty &= \frac{1}{2}\|g_h\|_\infty \leq \frac{1}{2}\|h'_1\|_\infty = \|h'\|_\infty, \\ \|f'_h\|_\infty &= \frac{1}{4}\|g'_h\|_\infty \leq \frac{1}{4} \cdot 2 \max\left(1, \frac{2}{\nu}\right)\|h'_1\|_\infty = \max\left(1, \frac{2}{\nu}\right)\|h'\|_\infty \quad \text{and} \\ \|f''_h\|_\infty &= \frac{1}{8}\|g''_h\|_\infty \leq \frac{1}{8}\left(2\|h''_1\|_\infty + 4 \max\left(1, \frac{2}{\nu}\right)\|h'_1\|_\infty\right) \\ &= \|h''\|_\infty + \max\left(1, \frac{2}{\nu}\right)\|h'\|_\infty. \end{aligned}$$

□

2.3. Exchangeable pairs. Let W, W' be identically distributed real-valued random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}[W^2] < \infty$. Assume that \mathcal{G} is a sub- σ -field of \mathcal{F} such that $\sigma(W) \subseteq \mathcal{G}$. Given a real number $\lambda > 0$ we define the random variables R and S via the *regression equations*

$$(2.9) \quad \frac{1}{\lambda}\mathbb{E}[W' - W \mid \mathcal{G}] = -W + R \quad \text{and}$$

$$(2.10) \quad \frac{1}{2\lambda}\mathbb{E}[(W' - W)^2 \mid \mathcal{G}] = 2(W + \nu) + S.$$

In many cases of interest Equation (2.9) holds with $R = 0$ for some (unique) $\lambda > 0$ but as was exemplified in [RR97] it is convenient to allow for a non-trivial remainder term R , in general. From Proposition 3.19 and Remark 3.10 of [Döb15], as well as from the bounds given by Theorem 2.3 we obtain the following new plug-in result for centered Gamma approximation. This theorem will play a major role in the proof of Theorem 1.1.

THEOREM 2.5. *Let W and W' be as above and assume that h is continuously differentiable on \mathbb{R} such that both h and h' are Lipschitz-continuous. Then, we have the bound*

$$\begin{aligned} \left| \mathbb{E}[h(W)] - \mathbb{E}[h(Z_\nu)] \right| &\leq \|h'\|_\infty (\max(1, 2\nu^{-1})\mathbb{E}|S| + \mathbb{E}|R|) \\ &\quad + \frac{\max(1, \frac{2}{\nu})\|h'\|_\infty + \|h''\|_\infty}{6\lambda} \mathbb{E}|W' - W|^3. \end{aligned}$$

If, moreover, $\mathbb{E}[W^2] = 2\nu$ and (2.9) holds with $R = 0$, then, since $\mathbb{E}[S] = 0$ in this case, we also have the bound

$$(2.11) \quad \begin{aligned} \left| \mathbb{E}[h(W)] - \mathbb{E}[h(Z_\nu)] \right| &\leq \max\left(1, \frac{2}{\nu}\right) \|h'\|_\infty \sqrt{\text{Var}(S)} \\ &\quad + \frac{\max\left(1, \frac{2}{\nu}\right) \|h'\|_\infty + \|h''\|_\infty}{6\lambda} \mathbb{E}|W' - W|^3. \end{aligned}$$

2.4. Proofs. The following two lemmas will be needed for the proof of Theorem 2.1.

LEMMA 2.6. *Let h be a Borel-measurable function h on \mathbb{R} with $\mathbb{E}|h(X_{r,\lambda})| < \infty$. Then, on each of the two intervals $(-\infty, 0)$ and $(0, \infty)$, there exists at most one bounded solution f of (1.2).*

Proof. Let f be a bounded solution of (1.2) on $(-\infty, 0)$. The solutions of the corresponding homogeneous equation are given by the constant multiples of the function

$$\psi(x) := |x|^{-r} e^{\lambda x}, \quad x < 0.$$

Thus, if g is another solution of (1.2) on $(-\infty, 0)$, then there is a constant $c \in \mathbb{R}$ such that

$$g = f + c\psi.$$

As $\psi(0-) = -\infty$ and $\sup_{x < 0} |f(x)| < \infty$, it follows that g can only be bounded if $c = 0$, i.e. if $g = f$. The proof for the interval $(0, \infty)$ is very similar. \square

LEMMA 2.7. *Let $a < b$ be real numbers and let $f : [a, b] \rightarrow \mathbb{R}$ be a function having the following properties:*

- (a) *f is continuous on $[a, b]$.*
- (b) *$f|_{[c,b]}$ is absolutely continuous for each $a < c < b$ (and, hence, f is λ -almost everywhere differentiable on $(a, b]$).*
- (c) *There is some $a < d < b$, a set $A \subseteq (a, d)$ at each of whose points f is differentiable with $\lambda((a, d) \setminus A) = 0$ and a real number γ such that $\lim_{n \rightarrow \infty} f'(x_n) = \gamma$ for each sequence $(x_n)_{n \in \mathbb{N}}$ lying in A with $\lim_{n \rightarrow \infty} x_n = a$.*

Then, f is absolutely continuous on $[a, b]$ and differentiable at a with $f'(a) = \gamma$. Furthermore, the function f' restricted to $A \cup \{a\}$ is continuous at a .

Proof. A proof can be found in the appendix of the thesis [Döb12a]. \square

We are now in the position to prove Theorem 2.1.

Proof of Theorem 2.1. It suffices to prove the theorem for the case $\lambda = 1$. In fact, it is easy to see that if g solves

$$xg'(x) + (r - x)g(x) = h_1(x) - \mathbb{E}[h_1(X_{r,1})],$$

where

$$h_1(x) := h(x/\lambda),$$

then $f(x) := g(\lambda x)$ solves (1.2). Taking into account the identities

$$f^{(k)}(x) = \lambda^k g^{(k)}(\lambda x), \quad \|h'_1\|_\infty = \lambda^{-1} \|h'\|_\infty \quad \text{and} \quad \|h''_1\|_\infty = \lambda^{-2} \|h''\|_\infty$$

then yields the bounds for general $\lambda > 0$. So let us assume for the rest of the proof that $\lambda = 1$. For notational convenience we will also write p_r for $p_{r,1}$, F_r for $F_{r,1}$ and X_r for $X_{r,1}$.

We first prove (a). As h is continuous, it is known (see e.g. [Döb15], Proposition 3.8) that the function $f_h : (0, \infty) \rightarrow \mathbb{R}$ with

$$(2.12) \quad \begin{aligned} f_h^+(x) &:= \frac{1}{xp_r(x)} \int_0^x (h(t) - \mathbb{E}[h(X_r)]) p_r(t) dt \\ &= -\frac{1}{xp_r(x)} \int_x^\infty (h(t) - \mathbb{E}[h(X_r)]) p_r(t) dt \end{aligned}$$

is a continuously differentiable solution to (1.2) on $(0, \infty)$ which can be continuously extended to $[0, \infty)$ by letting

$$(2.13) \quad f_h^+(0) := \frac{h(0) - \mathbb{E}[h(X_{r,1})]}{r}.$$

For a Lipschitz-continuous test function h we know from Corollary 3.15 of [Döb15] that $\sup_{x \geq 0} |f_h^+(x)| \leq \|h'\|_\infty$ and that for each $x > 0$ we have

$$|(f_h^+)'(x)| \leq 2\|h'\|_\infty \frac{\int_0^x F_r(t) dt \int_x^\infty (1 - F_r(t)) dt}{x^2 p_r(x)} =: 2\|h'\|_\infty S_r(x).$$

We bound $S_r(x)$ for $0 < x \leq r$ and for $x > r$ separately. Assume $x > r$. From Fubini's theorem we conclude that

$$\begin{aligned} \int_x^\infty (1 - F_r(t)) dt &= \int_x^\infty \int_t^\infty p_r(s) ds dt = \int_x^\infty (s - x) p_r(s) ds \\ &= \int_x^\infty (s - r) p_r(s) ds + (r - x)(1 - F_r(x)) \\ &= xp_r(x) + (r - x)(1 - F_r(x)) \leq xp_r(x). \end{aligned}$$

Also note that

$$(2.14) \quad \int_0^x F_r(t) dt \leq xF_r(x), \quad x \geq 0,$$

as F_r is nondecreasing. Hence, we obtain that

$$S_r(x) \leq \frac{x F_r(x) \cdot x p_r(x)}{x^2 p_r(x)} = F_r(x) \leq 1, \quad r < x.$$

Now let $0 < x \leq r$. Note first that

$$(2.15) \quad \int_x^\infty (1 - F_r(t)) dt \leq \int_0^\infty (1 - F_r(t)) dt = \mathbb{E}[X_{r,1}] = r.$$

Next, consider

$$\begin{aligned} R(x) &:= \frac{\int_0^x F_r(t) dt}{x^2 p_r(x)} = \frac{x p_r(x) - (r - x) F_r(x)}{x^2 p_r(x)} \quad \text{with} \\ R'(x) &= \frac{x F_r(x) - (1 + r - x) \int_0^x F_r(t) dt}{x^3 p_r(x)} =: \frac{N(x)}{x^3 p_r(x)}. \end{aligned}$$

We claim that $N(x) \geq 0$ for all $x \geq 0$, which implies that R is increasing. Note first that $N(0) = 0$. Also, we have

$$\begin{aligned} N'(x) &= F_r(x) + xp_r(x) + \int_0^x F_r(t)dt - (1+r-x)F_r(x) \\ &= xp_r(x) - (r-x)F_r(x) + \int_0^x F_r(t)dt \\ &= 2 \int_0^x F_r(t)dt \geq 0. \end{aligned}$$

Hence, N is increasing and, thus, $N(x) \geq 0$ for all $x \geq 0$. This implies that

$$\sup_{0 < x \leq r} R(x) = R(r) = \frac{1}{r}$$

and from (2.15) we conclude that

$$\sup_{0 < x \leq r} S_r(x) \leq r \sup_{0 < x \leq r} R(x) = \frac{r}{r} = 1.$$

Thus, we have proved that

$$\sup_{x > 0} |(f_h^+)'(x)| \leq 2\|h'\|_\infty.$$

In order to solve (1.2) on $(-\infty, 0)$ we use the theory developed in Section 2.4 of [Döb12a] (see also the unpublished manuscript [Döb12b]). There it is shown that a solution to (1.2) on $(-\infty, 0)$ is given by

$$\begin{aligned} f_h^-(x) &= \exp(-G_l(x)) \int_0^x (h(t) - \mathbb{E}[h(X_r)]) \frac{\exp(G_l(t))}{t} dt \\ (2.16) \quad &= \frac{1}{xq_l(x)} \int_0^x (h(t) - \mathbb{E}[h(X_r)]) q_l(t) dt, \end{aligned}$$

where G_l is an arbitrary primitive function of $x \mapsto \frac{r-x}{x}$ on $(-\infty, 0)$ and

$$q_l(x) := \frac{\exp(G_l(x))}{x}, \quad x < 0.$$

Also f_h^- can be continuously extended to $(-\infty, 0]$ by letting

$$(2.17) \quad f_h^-(0) := \frac{h(0) - \mathbb{E}[h(X_r)]}{r}.$$

This follows from Proposition 2.4.28 of [Döb12a] (or Proposition 2.22 of [Döb12b]). Again, by the continuity of h , it is easy to see that f_h^- is continuously differentiable on $(-\infty, 0)$. We choose

$$G_l(x) := r \log(-x) - x, \quad x < 0,$$

yielding

$$q_l(x) = -(-x)^{r-1} e^{-x}, \quad x < 0.$$

Note that $q_l(x) < 0$ for all $x \in (-\infty, 0)$. Furthermore, we define the function Q_l on $(-\infty, 0)$ by

$$Q_l(x) := \int_0^x q_l(t)dt = \int_x^0 (-q_l(t))dt > 0.$$

By taking its derivative we see that Q_l is decreasing on $(-\infty, 0)$. From Corollary 2.4.36 in [Döb12a] (or Corollary 2.28 of [Döb12b]) we have

$$\sup_{x < 0} |f_h^-(x)| \leq \|h'\|_\infty$$

and, for each $x \in (-\infty, 0)$, that

$$\begin{aligned} |(f_h^-)'(x)| &\leq 2\|h'\|_\infty \frac{(r-x)\left(-xQ_l(x) + \int_0^x tq_l(t)dt\right)}{-x^2q_l(x)} \\ &= 2\|h'\|_\infty \frac{(r-x) \int_x^0 Q_l(t)dt}{-x^2q_l(x)} \\ (2.18) \quad &=: 2\|h'\|_\infty U(x). \end{aligned}$$

Define the function T as well as q_u and Q_u on $(0, \infty)$ by $T(y) := U(-y)$, $q_u(y) := -q_l(-y) = y^{r-1}e^y$ and

$$\begin{aligned} Q_u(y) &:= Q_l(-y) = \int_0^{-y} q_l(t)dt = -\int_0^y q_l(-s)ds \\ &= \int_0^y q_u(s)ds. \end{aligned}$$

Then, we have

$$\begin{aligned} \int_{-y}^0 Q_l(t)dt &= -\int_y^0 Q_l(-s)ds = \int_0^y Q_u(s)ds = \int_0^y \int_0^t q_u(t)dt ds \\ &= \int_0^y (y-t)q_u(t)dt = yQ_u(y) - \int_0^y tq_u(t)dt \end{aligned}$$

as well as the representations

$$(2.19) \quad T(y) = \frac{(r+y) \int_0^y Q_u(s)ds}{y^2q_u(y)} = \frac{(r+y) \int_0^y Q_u(s)ds}{y^{r+1}e^y}$$

$$(2.20) \quad = \frac{(r+y) \int_0^y (y-t)q_u(t)dt}{y^{r+1}e^y} = \frac{(r+y)(yQ_u(y) - \int_0^y tq_u(t)dt)}{y^{r+1}e^y}$$

Note that, using de l'Hôpital's rule, we obtain

$$\begin{aligned} \lim_{y \downarrow 0} T(y) &= r \lim_{y \downarrow 0} \frac{\int_0^y Q_u(s)ds}{y^{r+1}e^y} = r \lim_{y \downarrow 0} \frac{Q_u(y)}{y^r e^y (r+1+y)} \\ &= \frac{r}{r+1} \lim_{y \downarrow 0} \frac{Q_u(y)}{y^r e^y} = \frac{r}{r+1} \lim_{y \downarrow 0} \frac{y^{r-1}e^y}{y^{r-1}e^y (r+y)} \\ (2.21) \quad &= \frac{r}{r+1} \lim_{y \downarrow 0} \frac{1}{r+y} = \frac{1}{r+1} \end{aligned}$$

as well as

$$\begin{aligned} \lim_{y \rightarrow \infty} T(y) &= \lim_{y \rightarrow \infty} \frac{r+y}{y} \cdot \lim_{y \rightarrow \infty} \frac{\int_0^y Q_u(s)ds}{y^r e^y} = \lim_{y \rightarrow \infty} \frac{y^{r-1}e^r}{(r+y)y^{r-1}e^y} \\ (2.22) \quad &= \lim_{y \rightarrow \infty} \frac{1}{r+y} = 0. \end{aligned}$$

By the continuity of T , from (2.21) and (2.22) we already conclude that

$$(2.23) \quad \sup_{x < 0} U(x) = \sup_{y > 0} T(y) < \infty.$$

Hence, it remains to deal with the local maxima of the function T . Note that

$$\begin{aligned} T'(y) &= \frac{y(r+y)Q_u(y) + \int_0^y Q_u(s)ds (y - (r+1+y)(r+y))}{y^{r+2}e^y} \\ &= \frac{y(r+y)Q_u(y) - \int_0^y Q_u(s)ds (y^2 + 2ry + r^2 + r)}{y^{r+2}e^y}. \end{aligned}$$

If $y_0 \in (0, +\infty)$ is a locally maximal point of T , then $T'(y_0) = 0$. This implies that

$$\begin{aligned} \int_0^{y_0} Q_u(s)ds &= \frac{y_0(r+y_0)Q_u(y_0)}{y_0^2 + 2ry_0 + r^2 + r} \quad \text{and} \\ T(y_0) &= \frac{(r+y_0)^2 Q_u(y_0)}{y_0^r e^{y_0} (y_0^2 + 2ry_0 + r^2 + r)} \leq \frac{Q_u(y_0)}{y_0^r e^{y_0}}. \end{aligned}$$

Define the function T_2 on $(0, \infty)$ by

$$T_2(y) := \frac{Q_u(y)}{y^r e^y}.$$

Then, the above discussion as well as (2.21) and (2.22) show that

$$\begin{aligned} \sup_{y > 0} T(y) &\leq \max \left(\lim_{y \downarrow 0} T(y), \lim_{y \rightarrow \infty} T(y), \sup_{y > 0} T_2(y) \right) \\ (2.24) \quad &= \max \left(\frac{1}{r+1}, \sup_{y > 0} T_2(y) \right). \end{aligned}$$

Using the explicit expression of Q_u , as well as the elementary estimate $1 \leq e^s \leq e^y$ for every $0 \leq s \leq y$, one deduces immediately that

$$(2.25) \quad \frac{e^{-y}}{r} \leq T_2(y) \leq \frac{1}{r}, \quad y > 0.$$

Hence, from (2.18), (2.23) and (2.25) we conclude that for all $x \in (-\infty, 0)$ we have

$$(2.26) \quad |(f_h^-)'(x)| \leq \frac{2}{r} \|h'\|_\infty.$$

Now, we define the function f_h on \mathbb{R} by letting

$$f_h(x) := \begin{cases} f_h^-(x), & x \leq 0 \\ f_h^+(x), & x \geq 0. \end{cases}$$

Note that (2.13) and (2.17) imply that f_h is well-defined. As the continuous concatenation of two Lipschitz-continuous functions, we recognize f_h to be Lipschitz-continuous on \mathbb{R} with

$$\begin{aligned} \|f_h\|_\infty &\leq \max \left(\sup_{x \leq 0} |f_h^-(x)|, \sup_{x \geq 0} |f_h^+(x)| \right) \leq \|h'\|_\infty \quad \text{and} \\ \|f_h'\|_\infty &\leq \max \left(\sup_{x < 0} |(f_h^-)'(x)|, \sup_{x > 0} |(f_h^+)'(x)| \right) \leq \max \left(2, \frac{2}{r} \right) \|h'\|_\infty. \end{aligned}$$

This finishes the proof of (a).

To prove (b) we assume that h is continuously differentiable and that both h and its derivative h' are Lipschitz-continuous. The identity

$$xf'_h(x) + (r - x)f_h(x) = h(x) - \mathbb{E}[h(X_{r,1})]$$

implies that f_h is continuously differentiable on both of the two intervals $(-\infty, 0)$ and $(0, \infty)$ and differentiating yields that f'_h solves

$$(2.27) \quad xg'(x) + (r + 1 - x)g(x) = h'(x) + f_h(x) =: h_2(x)$$

on both intervals $(-\infty, 0)$ and $(0, \infty)$. Note that (2.27) is the Stein equation for the distribution $\Gamma(r + 1, 1)$ corresponding to the test function h_2 . We already know from part (a) that f'_h is bounded by $2 \max(1, r^{-1}) \|h'\|_\infty$. Also, Proposition 3.17 of [Döb15] implies that h_2 is centered with respect to the $\Gamma(r + 1, 1)$ distribution. Hence, as h_2 is Lipschitz-continuous with Lipschitz-constant

$$(2.28) \quad \|h'_2\|_\infty \leq \|h''\|_\infty + \|f'_h\|_\infty \leq \|h''\|_\infty + 2 \max(1, r^{-1}) \|h'\|_\infty$$

we know from part (a) applied to the distribution $\Gamma(r + 1, 1)$ that there is a bounded solution g_{h_2} of (2.27). Since f'_h is bounded on both of the intervals $(-\infty, 0)$ and $(0, \infty)$, it thus follows from Lemma 2.6 that $f'_h(x) = g_{h_2}(x)$ for all $x \neq 0$. Since g_{h_2} is continuous at 0 we know from the analogs of (2.13) and (2.17) for g_{h_2} that

$$\begin{aligned} \lim_{x \uparrow 0} f'_h(x) &= \lim_{x \uparrow 0} g_{h_2}(x) = g_{h_2}(0) = \lim_{x \rightarrow 0} g_{h_2}(x) = \frac{h_2(0) - \mathbb{E}[h_2(X_{r+1,1})]}{r + 1} = \frac{h_2(0)}{r + 1} \\ &= \frac{h'(0)}{r + 1} + \frac{h(0) - \mathbb{E}[h(X_{r,\lambda})]}{r(r + 1)}, \end{aligned}$$

and, similarly we conclude that

$$\lim_{x \downarrow 0} f'_h(x) = \lim_{x \downarrow 0} g_{h_2}(x) = g_{h_2}(0) = \frac{h'(0)}{r + 1} + \frac{h(0) - \mathbb{E}[h(X_{r,\lambda})]}{r(r + 1)}.$$

By Lemma 2.7 this implies that f_h is continuously differentiable on \mathbb{R} with $f'_h(0) = g_{h_2}(0)$. Hence, we have $f'_h = g_{h_2}$ and from part (a) and (2.28) we conclude that f'_h is Lipschitz-continuous with

$$\begin{aligned} \|f''_h\|_\infty &= \|g'_{h_2}\|_\infty \leq 2 \max\left(1, \frac{1}{r + 1}\right) \|h'_2\|_\infty = 2 \|h'_2\|_\infty \\ &\leq 2 \left(\|h''\|_\infty + 2 \max(1, r^{-1}) \|h'\|_\infty \right) \\ &= 2 \|h''\|_\infty + 4 \max(1, r^{-1}) \|h'\|_\infty. \end{aligned}$$

□

REMARK 2.8. As anticipated, the factor $2/r$ in (2.4) cannot be replaced by a quantity that is uniformly bounded in r . Indeed, from the proof of Proposition 2.4.35 in [Döb12a] we have the representation

$$f'_h(x) = \frac{1}{-x^2 q_l(x)} \left((x - r) \int_x^0 Q_l(s) h'(s) ds - \int_x^\infty (1 - F(s)) h'(s) ds \int_x^0 Q_l(t) dt \right),$$

whenever h is Lipschitz-continuous and $x < 0$. If we take $h(x) = \min(x, 0)$, then we obtain

$$\int_x^0 Q_l(s)h'(s)ds = \int_0^x Q_l(s)ds \quad \text{and} \\ - \int_x^\infty (1 - F(s))h'(s)ds \int_x^0 Q_l(t)dt = x \int_x^0 Q_l(t)dt$$

because $F(s) = 0$ for all $s \leq 0$. This gives

$$f'_h(x) = \frac{(2x - r) \int_x^0 Q_l(t)dt}{-x^2 q_l(x)}$$

and, hence, straightforward estimates yield that, for $x < 0$,

$$|f'_h(x)| = \frac{(r - 2x) \int_x^0 Q_l(t)dt}{-x^2 q_l(x)} \geq e^x \frac{r - 2x}{r(r + 1)}.$$

In particular, this implies that

$$\sup_{x < 0} |f'_h(x)| \geq |f'_h(-1/2)| \geq e^{-1/2} \frac{1}{r}.$$

By mollifying the Lipschitz function $h(x) = \min(x, 0)$, one can also construct a function $h^* \in \mathcal{H}_2$ such that $\sup_{x < 0} |f'_{h^*}(x)| \geq e^{-1/2} \frac{1}{2r}$. This justifies the remark following Theorem 1.1.

3. PROOF OF THEOREM 1.1

We are going to apply the bound (2.11) from Theorem 2.5 with $\|h'\|_\infty, \|h''\|_\infty \leq 1$ to the σ -field $\mathcal{G} = \sigma(X_1, \dots, X_n)$ and to the following exchangeable pair (W, W') which has already been used in [DP16]: Let $Y := (Y_j)_{1 \leq j \leq n}$ be an independent copy of $X := (X_j)_{1 \leq j \leq n}$ and let α be uniformly distributed on $\{1, \dots, n\}$ such that X, Y and α are jointly independent. Letting, for $j = 1, \dots, n$,

$$X'_j := \begin{cases} Y_j, & \text{if } \alpha = j \\ X_j, & \text{if } \alpha \neq j \end{cases}$$

and

$$X' := (X'_1, \dots, X'_n)$$

it is easy to see that the pair (X, X') is exchangeable. Finally, as exchangeability is preserved under functions, letting

$$\begin{aligned} W' &:= \psi(X'_1, \dots, X'_n) = \sum_{j=1}^n 1_{\{\alpha=j\}} \left(\sum_{J: j \notin J} W_J + \sum_{J: j \in J} W_J^{(j)} \right) \\ &=: \sum_{J: \alpha \notin J} W_J + \sum_{J: \alpha \in J} W_J^{(\alpha)}, \end{aligned}$$

the pair (W, W') is exchangeable. Here, for $J = \{j_1, \dots, j_m\} \subseteq [n]$ with $1 \leq j_1 < j_2 < \dots < j_m \leq n$ and $j = j_k \in J$, we write

$$W_J^{(j)} := \psi_J(X_{j_1}, \dots, X_{j_{k-1}}, Y_{j_k}, X_{j_{k+1}}, \dots, X_{j_m}).$$

From Lemma 2.2 of [DP16] we know that (2.9) holds for the pair (W, W') with $R = 0$ and $\lambda = d/n$. Also, if we denote by

$$W^2 = \sum_{\substack{M \subseteq [n]: \\ |M| \leq 2d}} U_M$$

the Hoeffding decomposition of W^2 , then Lemma 2.6 of [DP16] states that

$$(3.1) \quad \frac{n}{2d} \mathbb{E}[(W' - W)^2 \mid X] = \sum_{\substack{M \subseteq [n]: \\ |M| \leq 2d-1}} a_M U_M,$$

where

$$a_M := 1 - \frac{|M|}{2d}, \quad M \subseteq [n] \quad \text{such that } |M| \leq 2d.$$

Hence, we have the following Hoeffding decomposition (3.2) of S :

$$\begin{aligned} S &= \frac{n}{2d} \mathbb{E}[(W' - W)^2 \mid X] - 2(W + \nu) = \sum_{\substack{M \subseteq [n]: \\ |M| \leq 2d-1}} \left(1 - \frac{|M|}{2d}\right) U_M - 2W - 2\nu \\ &= \sum_{\substack{M \subseteq [n]: \\ 1 \leq |M| \leq 2d-1}} \left(1 - \frac{|M|}{2d}\right) U_M - 2 \sum_{J \in \mathcal{D}_d} W_J \\ (3.2) \quad &= \sum_{\substack{M \subseteq [n]: \\ 1 \leq |M| \leq 2d-1, \\ |M| \neq d}} \left(1 - \frac{|M|}{2d}\right) U_M + \frac{1}{2} \sum_{J \in \mathcal{D}_d} (U_J - 4W_J) \\ &=: S_1 + \frac{1}{2} S_2. \end{aligned}$$

Here we have used that $U_\emptyset = \mathbb{E}[W^2] = 2\nu$. By the orthogonality of the terms appearing in the Hoeffding decomposition we thus obtain that

$$\text{Var}(S) = \text{Var}(S_1) + \frac{1}{4} \text{Var}(S_2).$$

From the orthogonality of the Hoeffding decomposition, we conclude that

$$(3.3) \quad \mathbb{E}[W^3] = \sum_{J \in \mathcal{D}_d} \sum_{\substack{M \subseteq [n]: \\ |M| \leq 2d}} \mathbb{E}[W_J U_M] = \sum_{J \in \mathcal{D}_d} \mathbb{E}[W_J U_J] = \sum_{J \in \mathcal{D}_d} \text{Cov}(W_J, U_J).$$

Before we proceed, we need an auxiliary lemma which expresses the fourth moment of W in terms of the exchangeable pair (W, W') . We first state a more general lemma, whose statement is in fact only a slight generalization of one of the key relations in Stein's method of exchangeable pairs (see [Ste86] or [CGS11]) will be very useful.

LEMMA 3.1. *Let (W, W') be an exchangeable pair of real-valued random variables such that, for some $\lambda > 0$, (2.9) is satisfied with $R = 0$ and let g be an absolutely continuous function on \mathbb{R} with $\mathbb{E}[(1 + |W| + |W'|)|g(W)|] < \infty$. Then,*

$$\mathbb{E}[W g(W)] = \mathbb{E}\left[g'(W) \frac{1}{2\lambda} \mathbb{E}[(W' - W)^2 \mid \mathcal{G}]\right] + E,$$

where

$$E := \frac{1}{2\lambda} \mathbb{E} \left[(W' - W)^2 \int_0^1 \left(g'(W + t(W' - W)) - g'(W) \right) dt \right]$$

is a remainder term.

LEMMA 3.2. *Let (W, W') be an exchangeable pair of real-valued random variables in $L^4(\mathbb{P})$ such that, for some $\lambda > 0$, (2.9) is satisfied with $R = 0$. Then,*

$$(3.4) \quad \mathbb{E}[W^4] = 3\mathbb{E} \left[W^2 \frac{1}{2\lambda} \mathbb{E}[(W' - W)^2 | \mathcal{G}] \right] - \frac{1}{4\lambda} \mathbb{E}[(W' - W)^4] \quad \text{and}$$

$$(3.5) \quad \mathbb{E}[W^3] = 2\mathbb{E} \left[W \frac{1}{2\lambda} \mathbb{E}[(W' - W)^2 | \mathcal{G}] \right].$$

Proof. The proof of (3.4) applies Lemma 3.1 with $g(x) = x^3$ leading to the remainder term

$$\begin{aligned} E &= \frac{3}{2\lambda} \mathbb{E} \left[(W' - W)^2 \int_0^1 \left(2tW(W' - W) + t^2(W' - W)^2 \right) dt \right] \\ &= \frac{3}{2\lambda} \mathbb{E}[W(W' - W)^3] + \frac{1}{2\lambda} \mathbb{E}[(W' - W)^4]. \end{aligned}$$

By exchangeability we obtain that

$$\mathbb{E}[(W' - W)^4] = \mathbb{E}[W'(W' - W)^3] - \mathbb{E}[W(W' - W)^3] = -2\mathbb{E}[W(W' - W)^3]$$

yielding the claim. In order to prove (3.5) we apply Lemma 3.1 with $g(x) = x^2$ leading to the remainder term

$$E = \frac{1}{\lambda} \mathbb{E} \left[(W' - W)^2 \int_0^1 t(W' - W) dt \right] = \frac{1}{2\lambda} \mathbb{E}[(W' - W)^3] = 0$$

again by exchangeability. □

Now, using (3.4) we obtain

$$\begin{aligned} \mathbb{E}[W^4] &= 3\mathbb{E} \left[W^2 \frac{n}{2d} \mathbb{E}[(W' - W)^2 | X] \right] - \frac{n}{4d} \mathbb{E}[(W' - W)^4] \\ &= 3 \sum_{\substack{M, N \subseteq [n]: \\ |M|, |N| \leq 2d}} \left(1 - \frac{|M|}{2d} \right) \mathbb{E}[U_N U_M] - \frac{n}{4d} \mathbb{E}[(W' - W)^4] \\ (3.6) \quad &= 12\nu^2 + 3 \sum_{\substack{M \subseteq [n]: \\ 1 \leq |M| \leq 2d-1}} \left(1 - \frac{|M|}{2d} \right) \text{Var}(U_M) - \frac{n}{4d} \mathbb{E}[(W' - W)^4]. \end{aligned}$$

where we have used that $U_\emptyset = \mathbb{E}[W^2] = 2\nu$. We have

$$\begin{aligned} \text{Var}(S_2) &= \sum_{J \in \mathcal{D}_d} \text{Var}(U_J - 4W_J) = \sum_{J \in \mathcal{D}_d} (\text{Var}(U_J) + 16 \text{Var}(W_J) - 8 \text{Cov}(U_J, W_J)) \\ &= \sum_{J \in \mathcal{D}_d} \text{Var}(U_J) + 32\nu - 8 \sum_{J \in \mathcal{D}_d} \text{Cov}(U_J, W_J) \\ &= \sum_{J \in \mathcal{D}_d} \text{Var}(U_J) + 32\nu - 8\mathbb{E}[W^3], \end{aligned}$$

where the last equality holds by virtue of (3.3). Hence, we have

$$\begin{aligned}
 \text{Var}(S) &= \text{Var}(S_1) + \frac{1}{4} \text{Var}(S_2) \\
 (3.7) \quad &= \sum_{\substack{M \subseteq [n]: \\ 1 \leq |M| \leq 2d-1, \\ |M| \neq d}} \left(1 - \frac{|M|}{2d}\right)^2 \text{Var}(U_M) + \frac{1}{4} \sum_{J \in \mathcal{D}_d} \text{Var}(U_J) + 8\nu - 2\mathbb{E}[W^3].
 \end{aligned}$$

From (3.6) and (3.7), using

$$\left(1 - \frac{|M|}{2d}\right)^2 \leq \left(1 - \frac{|M|}{2d}\right) \quad \text{for all } M \subseteq [n] \text{ such that } |M| \leq 2d,$$

we see that

$$\begin{aligned}
 &\mathbb{E}[W^4] - 12\mathbb{E}[W^3] - 12\nu^2 + 48\nu \\
 &= 3 \sum_{\substack{M \subseteq [n]: \\ 1 \leq |M| \leq 2d-1}} \left(1 - \frac{|M|}{2d}\right) \text{Var}(U_M) - 12\mathbb{E}[W^3] + 48\nu - \frac{n}{4d} \mathbb{E}[(W' - W)^4] \\
 &= 3 \sum_{\substack{M \subseteq [n]: \\ 1 \leq |M| \leq 2d-1, \\ |M| \neq d}} \left(1 - \frac{|M|}{2d}\right) \text{Var}(U_M) + \frac{3}{2} \sum_{J \in \mathcal{D}_d} \text{Var}(U_J) - 12\mathbb{E}[W^3] + 48\nu \\
 &\quad - \frac{n}{4d} \mathbb{E}[(W' - W)^4] \\
 &\geq 3 \left(\text{Var}(S_1) + \frac{1}{2} \left(\sum_{J \in \mathcal{D}_d} \text{Var}(U_J) - 8\mathbb{E}[W^3] + 32\nu \right) \right) - \frac{n}{4d} \mathbb{E}[(W' - W)^4] \\
 &= 3 \left(\text{Var}(S_1) + \frac{1}{2} \text{Var}(S_2) \right) - \frac{n}{4d} \mathbb{E}[(W' - W)^4] \\
 &\geq 3 \text{Var}(S) - \frac{n}{4d} \mathbb{E}[(W' - W)^4].
 \end{aligned}$$

Hence, we obtain that

$$(3.8) \quad \text{Var}(S) \leq \frac{1}{3} \left(\mathbb{E}[W^4] - 12\mathbb{E}[W^3] - 12\nu^2 + 48\nu \right) + \frac{n}{12d} \mathbb{E}[(W' - W)^4]$$

and it thus remains to find a bound on

$$\frac{n}{d} \mathbb{E}[(W' - W)^4].$$

From the definition of the coupling (W, W') and by the inequality $(a + b)^4 \leq 8(a^4 + b^4)$ we conclude that

$$\begin{aligned}
 \mathbb{E}|W' - W|^4 &= \mathbb{E} \left| \sum_{J \in \mathcal{D}_d: \alpha \in J} (W_J^{(\alpha)} - W_J) \right|^4 = \frac{1}{n} \sum_{j=1}^n \mathbb{E} \left[\left(\sum_{J \in \mathcal{D}_d: j \in J} (W_J^{(j)} - W_J) \right)^4 \right] \\
 &\leq \frac{8}{n} \sum_{j=1}^n \mathbb{E} \left[\left(\sum_{J \in \mathcal{D}_d: j \in J} W_J^{(j)} \right)^4 + \left(\sum_{J \in \mathcal{D}_d: j \in J} W_J \right)^4 \right] \\
 &= \frac{16}{n} \sum_{j=1}^n \mathbb{E} \left[\left(\sum_{J \in \mathcal{D}_d: j \in J} W_J \right)^4 \right] = \frac{16}{n} \sum_{j=1}^n \sum_{\substack{J, K, L, M \in \mathcal{D}_d: \\ j \in J \cap K \cap L \cap M}} \mathbb{E}[W_J W_K W_L W_M] \\
 (3.9) \quad &= \frac{16}{n} \sum_{\substack{(J, K, L, M) \in \mathcal{D}_d^4: \\ J \cap K \cap L \cap M \neq \emptyset}} |J \cap K \cap L \cap M| \mathbb{E}[W_J W_K W_L W_M].
 \end{aligned}$$

Here, we have used the fact that the sums

$$\sum_{J \in \mathcal{D}_d: j \in J} W_J^{(j)} \quad \text{and} \quad \sum_{J \in \mathcal{D}_d: j \in J} W_J$$

are identically distributed for each $j \in [n]$. Now, by the definition of D and by the generalized Hölder inequality, for each $(J, K, L, M) \in \mathcal{D}_d^4$ we have

$$\begin{aligned}
 |\mathbb{E}[W_J W_K W_L W_M]| &\leq \left(\mathbb{E}[W_J^4] \mathbb{E}[W_K^4] \mathbb{E}[W_L^4] \mathbb{E}[W_M^4] \right)^{1/4} \\
 &\leq \left(D \sigma_J^4 D \sigma_K^4 D \sigma_L^4 D \sigma_M^4 \right)^{1/4} \leq D \sigma_J \sigma_K \sigma_L \sigma_M.
 \end{aligned}$$

Proposition 2.9 of [DP16] implies that

$$\sum_{\substack{(J, K, L, M) \in \mathcal{D}_d^4: \\ J \cap K \cap L \cap M \neq \emptyset}} \sigma_J \sigma_K \sigma_L \sigma_M \leq C_d \varrho_n^2,$$

where the finite constant C_d only depends on d . Thus, from (3.9) we conclude that

$$(3.10) \quad \mathbb{E}|W' - W|^4 \leq \frac{16}{n} C_d D_n \varrho_n^2.$$

From (3.8) and (3.10) we have

$$(3.11) \quad \text{Var}(S) \leq \frac{1}{3} \left| \mathbb{E}[W^4] - 12\mathbb{E}[W^3] - 12\nu^2 + 48\nu \right| + \frac{4}{3d} C_d D_n \varrho_n^2.$$

Also, from the fact that

$$\mathbb{E}[(W' - W)^2] = 2\lambda \mathbb{E}[W^2] = \frac{4d\nu}{n}$$

and using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
 \frac{1}{6\lambda} \mathbb{E}|W' - W|^3 &\leq \frac{n}{6d} \left(\mathbb{E}[(W' - W)^2] \right)^{1/2} \left(\mathbb{E}|W' - W|^4 \right)^{1/2} \\
 &= \frac{\sqrt{\nu}}{3\sqrt{d}} \left(n \mathbb{E}|W' - W|^4 \right)^{1/2} \leq \frac{\sqrt{\nu}}{3\sqrt{d}} \sqrt{16C_d D_n \varrho_n^2} \\
 (3.12) \quad &= \frac{4\sqrt{\nu}}{3\sqrt{d}} \sqrt{C_d D_n \varrho_n^2},
 \end{aligned}$$

where we have used (3.10) again. Theorem 1.1 now follows from (2.11), (3.11) and (3.12).

4. PROOF OF THEOREM 1.4

For the sake of completeness, we will discuss some further details concerning stochastic analysis for functionals of a Poisson measure. Throughout the section, we work in the same framework as the one outlined in Section 1.3.

For an integer $p \geq 1$ we denote by $L^2(\mu^p)$ the Hilbert space of all square-integrable real-valued functions on \mathcal{Z}^p and we write $L_s^2(\mu^p)$ for the subspace of those functions in $L^2(\mu^p)$ which are μ^p -a.e. symmetric. Moreover, for ease of notation, we denote by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ the usual norm and scalar product on $L^2(\mu^p)$ for whatever value of p . We further define $L^2(\mu^0) := \mathbb{R}$. For $f \in L^2(\mu^p)$ we denote by $I_p(f)$ the *multiple Wiener-Itô integral* of f with respect to $\hat{\eta}$. If $p = 0$, then, by convention, $I_0(c) := c$ for each $c \in \mathbb{R}$. The following properties of multiple integrals are standard for all $p, q \geq 0$:

- 1) $I_p(f) = I_p(\tilde{f})$, where \tilde{f} denotes the *canonical symmetrization* of $f \in L^2(\mu^p)$.
- 2) $I_p(f) \in L^2(\mathbb{P})$.
- 3) $\mathbb{E}[I_p(f)I_q(g)] = \delta_{p,q} p! \langle \tilde{f}, \tilde{g} \rangle$, where $\delta_{p,q}$ denotes *Kronecker's delta symbol*.

For $p \geq 0$, the Hilbert space consisting of all random variables $I_p(f)$, $f \in L^2(\mu^p)$, is called the *p-th Wiener chaos* associated with η . It is a crucial fact that every $F \in L^2(\mathbb{P})$ admits a unique representation

$$(4.1) \quad F = \mathbb{E}[F] + \sum_{p=1}^{\infty} I_p(f_p),$$

where $f_p \in L_s^2(\mu^p)$, $p \geq 1$, are suitable symmetric kernel functions. Identity (4.1) is called the *chaotic decomposition* of the functional $F \in L^2(\mathbb{P})$. Next, we briefly introduce the necessary Malliavin operators. The *domain* $\text{dom } D$ of the Malliavin derivative operator D is the set of all $F \in L^2(\mathbb{P})$ such that the chaotic decomposition (4.1) of F satisfies $\sum_{p=1}^{\infty} p p! \|f_p\|^2 < \infty$. For such an F the random function $\mathcal{Z} \ni z \mapsto D_z F \in L^2(\mathbb{P})$ is defined via

$$D_z F = \sum_{p=1}^{\infty} p I_{p-1}(f_p(z, \cdot)),$$

where $f_p(z, \cdot)$ is an a.e. symmetric function on \mathcal{Z}^{p-1} . Hence, $DF = (D_z F)_{z \in \mathcal{Z}}$ can be viewed as an element of $L^2(\Omega \times \mathcal{Z}, \mathcal{F} \otimes \mathcal{Z}, \mathbb{P} \otimes \mu)$. Note that, as $\mathcal{F} = \sigma(\eta)$, each $F \in L^2(\mathbb{P})$ can be written as $F = g(\eta)$ for some measurable functional g . Then, for

$z \in \mathcal{Z}$ we write $F_z := g(\eta + \delta_z)$. If, furthermore, F happens to be in $\text{dom } D$, then it is known that for μ -almost every $z \in \mathcal{Z}$ we have the important formula

$$(4.2) \quad D_z F = F_z - F.$$

The domain $\text{dom } L$ of the *Ornstein-Uhlenbeck generator* L is the set of those $F \in L^2(\mathbb{P})$ whose chaotic decomposition (4.1) verifies $\sum_{p=1}^{\infty} p^2 \|f_p\|^2 < \infty$ and, for $F \in \text{dom } L$, one defines

$$LF = - \sum_{p=1}^{\infty} p I_p(f_p).$$

By definition, $\mathbb{E}[LF] = 0$. The domain $\text{dom } L^{-1}$ of the *pseudo-inverse* L^{-1} of L is the class of mean zero elements F of $L^2(\mathbb{P})$. If $F = \sum_{p=1}^{\infty} I_p(f_p)$ is the chaotic decomposition of such an F , then it is defined via

$$L^{-1}F = \sum_{p=1}^{\infty} \frac{1}{p} I_p(f_p).$$

Note that these definitions imply that

$$LL^{-1}F = F \quad \text{for all } F \in \text{dom } L^{-1} \quad \text{and} \quad L^{-1}LF = F - \mathbb{E}[F] \quad \text{for all } F \in \text{dom } L.$$

Finally, we review the definition *Skohorod integral operator* δ . Note that for each $u \in L^2(\Omega \times \mathcal{Z}, \mathcal{F} \otimes \mathcal{Z}, \mathbb{P} \otimes \mu)$ and each fixed $z \in \mathcal{Z}$ we have a chaotic decomposition of the type

$$(4.3) \quad u_z = \sum_{p=0}^{\infty} I_p(f_p(z, \cdot))$$

and, for $p \geq 0$, the kernel $f_p(z, \cdot)$ is an element of $L_s^2(\mu^p)$. Then, the domain $\text{dom } \delta$ consists of those such $u \in L^2(\Omega \times \mathcal{Z}, \mathcal{F} \otimes \mathcal{Z}, \mathbb{P} \otimes \mu)$ whose kernels given by (4.3) satisfy

$$\sum_{p=0}^{\infty} (p+1)! \|f_p\|_{L^2(\mu^{p+1})}^2 < \infty$$

and, for $u \in \text{dom } \delta$, one lets

$$\delta(u) = \sum_{p=0}^{\infty} I_{p+1}(f_p).$$

The following two identities are essential for the Malliavin-Stein method on the Poisson space. The first one, the *integration by parts formula*, characterizes δ as the adjoint operator of D :

$$(4.4) \quad \mathbb{E}[G\delta(u)] = \mathbb{E}[\langle DG, u \rangle_{L^2(\mu)}] \quad \text{for all } G \in \text{dom } D, u \in \text{dom } \delta.$$

$$(4.5) \quad \delta DF = -LF \quad \text{for all } F \in \text{dom } L.$$

We are now in the position to prove our new bounds on the Gamma approximation for functionals on the Poisson space.

Proof of Theorem 1.4. The proof follows the lines of the proof of Theorem 3.1 of [PSTU10] very closely. Fix $h \in \mathcal{H}_2$ and write $f = f_h$ for the solution to the Stein

equation (2.7) from Theorem 2.3. Using the fact that $\mathbb{E}[F] = 0$ as well as (4.4) and (4.5) we have

$$\begin{aligned} \mathbb{E}[Ff(F)] &= \mathbb{E}[(LL^{-1}F)f(F)] = \mathbb{E}[-\delta(DL^{-1}F)f(F)] \\ (4.6) \quad &= \mathbb{E}[\langle Df(F), -DL^{-1}F \rangle_{L^2(\mu)}]. \end{aligned}$$

Now, for fixed $z \in \mathcal{Z}$, using (4.2) as well as Taylor's formula, we have

$$\begin{aligned} D_z f(F) &= (f(Z))_z - f(F) = f(F_z) - f(F) \\ (4.7) \quad &= f'(F)(F_z - F) + R(F_z - F) = f'(F)(D_z F) + R(D_z F), \end{aligned}$$

where $y \mapsto R(y)$ is a function which satisfies

$$|R(y)| \leq \frac{\|f''\|_\infty}{2} y^2 \leq \frac{\max(1, \frac{2}{\nu}) \|h'\|_\infty + \|h''\|_\infty}{2} y^2 \leq \max\left(1, \frac{1}{\nu} + \frac{1}{2}\right) y^2$$

by Theorem 2.3 (b). Hence, from (4.6) and (4.7) we conclude that

$$\mathbb{E}[Ff(F)] = \mathbb{E}[f'(F)\langle DF, -DL^{-1}F \rangle_{L^2(\mu)}] + \mathbb{E}[\langle R(DF), -DL^{-1}F \rangle_{L^2(\mu)}]$$

yielding

$$\begin{aligned} |\mathbb{E}[h(F)] - \mathbb{E}[h(Z_\nu)]| &= |\mathbb{E}[2(F + \nu)f'(F) - Ff(F)]| \\ &\leq |\mathbb{E}[f'(F)(2(F + \nu) - \langle DF, -DL^{-1}F \rangle_{L^2(\mu)})]| + |\mathbb{E}[\langle R(DF), -DL^{-1}F \rangle_{L^2(\mu)}]| \\ &\leq \max\left(1, \frac{2}{\nu}\right) \mathbb{E}|2(F + \nu) - \langle DF, -DL^{-1}F \rangle_{L^2(\mu)}| + \int_{\mathcal{Z}} \mathbb{E}|R(D_z F)D_z L^{-1}F| \mu(dz) \\ &\leq \max\left(1, \frac{2}{\nu}\right) \mathbb{E}|2(F + \nu) - \langle DF, -DL^{-1}F \rangle_{L^2(\mu)}| \\ &\quad + \max\left(1, \frac{1}{\nu} + \frac{1}{2}\right) \int_{\mathcal{Z}} \mathbb{E}|D_z F|^2 |D_z L^{-1}F| \mu(dz), \end{aligned}$$

which in turn gives (1.8). Applying Cauchy-Schwarz on (1.8) gives (1.9). The bounds (1.10) and (1.11) easily follow from these by the definitions of the Malliavin operators. \square

5. PROOFS OF THEOREM 1.6 AND OF PROPOSITION 1.7

Proof of Theorem 1.6. We have to show that, for every 1-Lipschitz test function h , the quantity $|\mathbb{E}[h(F)] - \mathbb{E}[h(Z_\nu)]|$ is bounded by the right-hand side of (1.12). We start by assuming that h is twice continuously differentiable and such that $\|h'\|_\infty \leq 1$. Then, we can use Theorem 2.3 to deduce that there exists a solution f_h to (2.7) such that f_h is continuously differentiable, and $\|f'_h\|_\infty \leq \max(1, \frac{2}{\nu}) \|h'\|_\infty$. It follows that

$$|\mathbb{E}[h(F)] - \mathbb{E}[h(Z_\nu)]| = \left| \mathbb{E}\left[f'_h(F) \mathbb{E}\{2(F + \nu) - \langle DF, -DL^{-1}F \rangle_{\mathcal{H}} \mid F\}\right] \right|,$$

where we have applied the standard integration by parts formula

$$\mathbb{E}[Ff_h(F)] = \mathbb{E}[f'_h(F)\langle DF, -DL^{-1}F \rangle_{\mathcal{H}}],$$

as well as the definition of conditional expectation. Observe that, in view of the smoothness of f_h , such an integration by parts relation holds for any $F \in \mathbb{D}^{1,2}$, irrespective of the fact that F has a density. To deal with a general 1-Lipschitz function h , one simply observes that there exists a family $\{h_\varepsilon : \varepsilon > 0\}$ of functions of class C^2 such that: (1) for each ε the first and second derivatives of h_ε are bounded,

(2) $\|h'_\varepsilon\|_\infty \leq \|h'\|_\infty$, and (3) $\|h - h_\varepsilon\|_\infty \rightarrow 0$, as $\varepsilon \rightarrow 0$ (one can take for example $h_\varepsilon(x) = \mathbb{E}[h(x + \varepsilon N)]$, where N is a standard normal random variable). \square

Proof of Proposition 1.7. The fact that (1.14) implies that F_n converges in distribution to Z_ν follows from Theorem 1.6, whereas the estimate (1.15) is an immediate consequence of [NP13, Theorem 3.1] and of the fact that the Fortet-Mourier distance is bounded (by definition) by d_1 . To conclude, we have to show that, if F_n converges in distribution to Z_ν , then (1.14) is necessarily verified. In order to do that, one can reason exactly as in the proof of [APP15, Theorem 3] and deduce that, if F_n converges in distribution to Z_ν , then, as $n \rightarrow \infty$ and for every fixed $M \in (0, \infty)$,

$$\sup_{\varphi \in \mathcal{F}_M} \mathbb{E} [\varphi(F_n)(2(F_n + \nu) - \langle DF_n, -DL^{-1}F_n \rangle_{\mathcal{H}})] \rightarrow 0,$$

where \mathcal{F}_M denotes the class of all Borel functions that are bounded by 1, and with support contained in $[-M, M]$. It follows that

$$\begin{aligned} & \mathbb{E} [|\mathbb{E} \{2(F_n + \nu) - \langle DF_n, -DL^{-1}F_n \rangle_{\mathcal{H}} \mid F_n\}|] \\ &= \sup_{\|\varphi\|_\infty \leq 1} \mathbb{E} [\varphi(F_n) \mathbb{E} \{2(F_n + \nu) - \langle DF_n, -DL^{-1}F_n \rangle_{\mathcal{H}} \mid F_n\}] \\ &\leq \sup_{\varphi \in \mathcal{F}_M} \mathbb{E} [\varphi(F_n) (2(F_n + \nu) - \langle DF_n, -DL^{-1}F_n \rangle_{\mathcal{H}})] \\ &\quad + \sqrt{\mathbb{P}[|F_n| > M]} \times \sup_k \mathbb{E} [(2(F_k + \nu) - \langle DF_k, -DL^{-1}F_k \rangle_{\mathcal{H}})^2]^{1/2}, \end{aligned}$$

and the conclusion is obtained by first letting $n \rightarrow \infty$, and next letting $M \rightarrow \infty$, where one has to use the fact that

$$\sup_k \mathbb{E} [(2(F_k + \nu) - \langle DF_k, -DL^{-1}F_k \rangle_{\mathcal{H}})^2]^{1/2} < \infty,$$

by virtue of the usual hypercontractivity properties enjoyed by random variables living in a finite sum of Wiener chaoses — see e.g. [NP12, Corollary 2.8.15]. \square

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